Euclidean Fairness and Efficiency

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The inherent vice of capitalism is the unequal sharing of blessings.
The inherent virtue of Socialism is the equal sharing of miseries.
Winston Churchill

Fairness and efficiency are often irreconcilable. Economists have long been repeating this. Indeed, one scholar, despite reading this article twice, still did not find anything in it to be surprising, curious or funny.[1] The ancients knew it all too well, as shown by Solomon's fair division of the baby. Yet, this basic fact of life is still unclear to many a wise man.

Ptolemy's Dilemma. The problem we are supposing may be most completely given in the form of the one that is said to have haunted Ptolemy I, King of Egypt. He wished to construct his Temple of the Muses (the famous Library) in the city of Alexandria. Alexandria had three neighborhoods along its coast: Rhakotis, the Jewish Quarter, and the Port, as shown by the map in Figure 1.

The inhabitants of each neighborhood wished the Temple to be built in their respective neighborhood. When Ptolemy summoned the wisest men of Egypt, they presented a fair solution: the Temple shall be built equally close to each neighborhood. It is at that time that Euclid presented the King with the manuscript we report below. It showed the King the location of the *fair* temple: a swamp, ten miles outside of Alexandria.

Not surprisingly, for those familiar with mathematical works of that age, the manuscript is dry. The figure therein has no obvious description or axes, perhaps because a Cartesian coordinate system was invented nineteen centuries after Euclid's work. The results in the manuscript are merely stated, with no intuition, no motivation, no technical footnotes, and no reference to empirical *stylized facts*. Previous literature is completely ignored too (though we argue this might be somewhat excusable). As a result, its implications might not be so apparent to our modern minds. "Ptolemy [himself] once asked [Euclid] if there was in geometry a way shorter than that of the elements; he replied that there was no royal road to geometry."[2]

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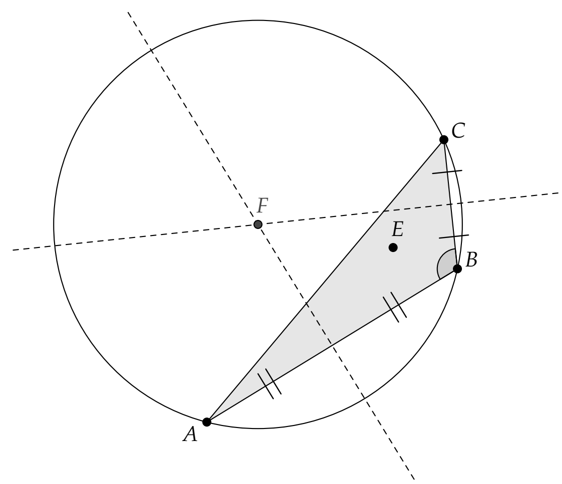
[1] We thank an anonymous referee for pointing this out to us.

[2] Proclus Diadochus, Commentary on Euclid's Elements, Book I, *Greek Mathematical Works, Volume I*, The Loeb Classical Library, 1939, London, W. Heinemann, p. 155.

Figure 1: Alexandria, 300 B.C.

The Manuscript



Three individuals have bliss policies A, B, and C that form a triangle.

Definition 1 (Fairness)**.** A policy F is fair if AF, BF, and CF are equal.

Definition 2 (Efficiency)**.** A policy E is efficient if it falls within the triangle ABC.

Notions of fairness and efficiency coincide with **utility equality** and **Pareto efficiency** if individual preferences are represented by Euclidean loss functions.

Proposition 1. *The fair policy is the center of the circle that circumscribes the triangle ABC.*

Proof. Follows from the definition of fairness and Euclid's Elements, Book IV, Proposition 5, *about a given triangle to circumscribe a circle*. □

Proposition 2. *The fair policy is efficient if and only if the triangle ABC is acute-angled.*

Proof. Follows from the definition of efficiency and Euclid's Elements, Book IV, Proposition 5, Porism, *that, when the center of the circle falls within the triangle, the angle ABC is less than a right angle; and when the center of the circle falls outside the triangle, the angle ABC is greater than a right angle.* □

Porism. *From this it is manifest that policies that are fair are not efficient in an **aligned society** (ABC is obtuse-angled).*

Afterthought

The problem in the manuscript corresponds to a spatial model of politics. Agents A, B, and C have preferences represented by quadratic loss functions over a bi-dimensional policy space:[3] if agent $i \in \{A, B, C\}$ has bliss point $b^i = (b^i_1, b^i_2)$, then her utility from policy $p = (p_1, p_2)$ is

$$u\left(p, b^i\right) = -\left(p_1 - b^i_1\right)^2 - \left(p_2 - b^i_2\right)^2.$$

Utility equality (what the manuscript refers to as *fairness*) is obtained with any policy p such that $u\left(p, b^A\right) = u\left(p, b^B\right) = u\left(p, b^C\right)$. Similarly, Pareto efficiency is obtained with any policy p that is a convex combination of b^A, b^B, and b^C. The manuscript uses a single theorem by Euclid to (i) characterize the (generically) unique fair policy, and (ii) determine necessary and sufficient conditions for the fair policy to be Pareto efficient.

The last porism in the manuscript suggests a further interpretation of Euclid's results: fairness is never efficient when the society is *aligned* in the sense that the agents disagree primarily along one out of two political dimensions. For example, let one dimension be economic issues and the other social issues. A society with heterogeneous social preferences and *little* economic disparity is *aligned*. A society with heterogeneous social preferences and *great* economic disparity is *misaligned*. The first would find it difficult to implement a policy that is both fair and efficient; the second would find it easy.

[3] The results could easily be extended to spaces of higher dimension and to non-quadratic loss functions.